

## Shifted $1/N$ expansion and exact solutions for the potential $V(r)=-Z/r+gr+\lambda r^2$

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## COMMENT

# Shifted $1/N$ expansion and exact solutions for the potential

$$V(r) = -Z/r + gr + \lambda r^2$$

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**Abstract.** We have found an infinite number of exact solutions for the hydrogenic atom in the external potential  $V(r) = gr + \lambda r^2$ , not only for an s-wave state but for higher waves as well, from supersymmetric considerations. The general Schrödinger equation has been treated by the shifted  $1/N$  expansion method. The eigenvalues obtained from the shifted  $1/N$  expansion are compared with those obtained by Bessis *et al* and also with the supersymmetric exact values.

## 1. Introduction

Recently Bessis *et al* (1987) obtained exact solutions for the states for the potential

$$V(r) = -Z/r + gr + \lambda r^2 \quad (1)$$

subject to special relations between  $g$ ,  $\lambda$  and the nuclear charge  $Z$ . Potential (1) describes a hydrogenic atom with the perturbation  $V(r) = gr + \lambda r^2$ . Potentials of the form  $V(r) = -Z/r + gr$  (linear plus Coulomb) have been studied extensively in the context of the quark model for the  $J/\psi$  spectrum and similar bound-state problems in particle physics (Quigg and Rosner 1979, Eichten *et al* 1978, Rein 1977). Killingbeck (1978) and Saxena and Varma (1982a) have studied the potential  $V(r) = -Z/r + 2\beta r + 2\beta^2 r^2$  with  $Z = 1$ . Analytical solutions for the s-wave states for some particular values of  $\beta$  were obtained by Saxena and Varma (1982b).

In the present comment we have obtained exact solutions for any value of the angular momentum when the coupling parameters satisfy certain relations among themselves. The solutions obtained by us are a generalisation of those obtained by Bessis *et al* (1987). We took a clue from the supersymmetric structure of the Hamiltonian for certain values of the parameters and then obtained the general solution for the wavefunctions in the form  $r^{l+1} \exp(-ar^2 - br) \prod_{i=1}^n (1 + g_i r)$ . For the general potential we have also applied the shifted  $1/N$  expansion method to obtain the energy eigenvalues. The shifted  $1/N$  expansion proposed by Sukhatme and Imbo (1983) differs from the ordinary large- $N$  expansion method by modifying the expansion parameter, which instead of  $k = N + 2l$  (Mlodinow and Papanicolaou 1980, 1981) becomes  $\bar{k} = N + 2l - a$ . The shift  $a$  is so chosen as to obtain exact analytical results for the harmonic oscillator. This method has proved to be a powerful method for obtaining the eigenvalues of spherically symmetric potentials (Dutt *et al* 1986a, b, Dutt and Varshni 1987, Roy and Roychoudhury 1987a, Varshni 1987). It is non-perturbative

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in nature (in terms of coupling parameters) and thus can be used in problems for large coupling parameters. The potential (1), for large  $Z$  and small  $g$  and  $\lambda$ , behaves like a Coulomb potential with a small perturbation and for large  $\lambda$  and relatively small  $Z$  and  $g$  behaves like a harmonic oscillator potential with small perturbation and hence it is suitable for treatment by the shifted  $1/N$  expansion method, especially for larger values of  $l$ . In § 2 we discuss the exact solutions obtained from supersymmetric considerations and in § 3 we present the shifted  $1/N$  expansion method for the potential (1). The results are discussed in § 4. Throughout the paper we shall use atomic units in which  $m = \hbar = c = 1$ .

## 2. Supersymmetric character of the hydrogenic atom with the perturbation $V(r) = gr + \lambda r^2$ and exact solutions

Before casting the problem in supersymmetric form we give below a summary of the salient features of supersymmetric quantum mechanics (SUSYQM) in one dimension. In one dimension the Hamiltonian of SUSYQM is given by

$$H^S = \{Q^\dagger, Q\} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (2)$$

where

$$H_\pm = -\frac{1}{2} d^2/dx^2 + V_\pm(x) \quad (3)$$

$$V_\pm(x) = \frac{1}{2}(W^2(x) \pm dW(x)/dx). \quad (4)$$

$W(x)$  is called the superpotential and  $Q, Q^\dagger$  the supercharges, whose explicit forms are given below:

$$Q = \frac{1}{\sqrt{2}}(p + iW) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (5)$$

$$Q^\dagger = \frac{1}{\sqrt{2}}(p - iW) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (6)$$

The relations obeyed by  $Q, Q^\dagger$  and  $H^S$  are the following:

$$[H^S, Q] = [H^S, Q^\dagger] = 0$$

$$Q^2 = Q^{\dagger 2} = 0.$$

The eigenstates of  $H^S$  are

$$\phi^n(x) = \begin{pmatrix} \phi_+^n(x) \\ \phi_-^n(x) \end{pmatrix}. \quad (7)$$

If supersymmetry is unbroken the ground-state energy is zero and the ground-state wavefunctions are of the form

$$\begin{pmatrix} \phi_+^0(x) \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \phi_-^0(x) \end{pmatrix} \quad (8)$$

depending on the normalisability of  $\phi_+^0(x)$  or  $\phi_-^0(x)$ . Now if  $|\psi\rangle$  is a ground state then

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = 0. \quad (9)$$

From (4) and (5) it follows that

$$\phi_{\pm}^0(x) = \exp\left(\pm \int^x W(t) dt\right). \quad (10)$$

Now consider the case of hydrogenic atom with a perturbation  $V(r) = gr + \lambda r^2$  with the Schrödinger equation given by

$$-\frac{1}{2} \frac{d^2\psi}{dr^2} + \left(\frac{l(l+1)}{2r^2} - \frac{Z}{r} + gr + \lambda r^2 - E\right)\psi = 0. \quad (11)$$

Following our previous method of constructing exact solutions of Schrödinger equation from supersymmetric considerations (Roy and Roychoudhury 1987b), the general ansatz for  $W$  is taken as

$$W = ar + b + \frac{c}{r} + \sum_{i=1}^n \frac{g_i}{1+g_i r}. \quad (12)$$

Then  $V_-(r) = \frac{1}{2}(W^2 - W')$  can be written in the form

$$\begin{aligned} 2V_-(r) = W^2 - W' &= \frac{c(c+1)}{r^2} + b^2 + a^2 r^2 + 2abr + \frac{2bc}{r} + 2ac \\ &- 2c \frac{\sum g_i}{r} + 2c \frac{\sum g_i^2}{1+g_i r} + \sum_{j=1}^n \sum_{\substack{i=1 \\ i \neq j}}^n \frac{g_i g_j}{g_i - g_j} \left( \frac{g_i}{1+g_i r} - \frac{g_j}{1+g_j r} \right) \\ &- 2b \sum \frac{g_i}{1+g_i r} - 2na + 2a \sum \frac{1}{1+g_i r} - a. \end{aligned} \quad (13)$$

Now the effective potential appearing in the Schrödinger equation (11) and that appearing corresponding to the potential  $V_-(r)$  in (13) are respectively

$$2V^{\text{eff}}(r) = 2\lambda r^2 + 2gr - \frac{2Z}{r} + \frac{l(l+1)}{r^2} \quad (14)$$

$$2V_-^{\text{eff}}(r) = 2V_-(r) + (2n+1-2c)a - b^2 \quad (15)$$

(this is  $V_-(r)$  without the constant term which can be absorbed in the definition of energy). Now we choose  $g_i$  and  $a$ ,  $b$  and  $c$  such that (14) and (15) become identical.

Comparing (14) and (15) we get

$$c = -(l+1) \quad a = (2\lambda)^{1/2} \quad b = g/a = g/(2\lambda)^{1/2}. \quad (16)$$

A negative value of  $c$  is taken so that  $\exp(-\int^x W dt)$  is normalisable and the  $g_i$  satisfy

$$2cg_i^2 - 2bg_i + 2a + 2g_i^2 \sum_{\substack{j=1 \\ j \neq i}}^n \frac{g_j}{g_i - g_j} = 0 \quad i = 0, 1, 2, \dots, n \quad \text{with } g_0 = 0 \quad (17)$$

and  $Z$  is given by

$$Z = c \sum g_i - bc. \quad (18)$$

The relation between the supersymmetric energy and the energy for the hydrogenic atom is obtained from the identity

$$2V_-(r) - 2E_- = 2V^{\text{eff}}(r) - 2E_h \quad (19a)$$

or

$$2E_- - (b^2 + 2ac - 2na - a) = 2E_h \quad (19b)$$

where  $E_h$  denotes the eigenvalues of the Schrödinger equation corresponding to the potential (14), and  $E_-$  is the eigenvalue corresponding to the potential  $\frac{1}{2}(W^2 - W')$ . From (19a) and (19b) we have

$$\begin{aligned} E_h &= E_- + \frac{1}{2}[(2n+1-2c)a - b^2] \\ &= E_- + (\frac{1}{2}\lambda)^{1/2}[(2n+3+2l)] - g^2/4\lambda. \end{aligned} \quad (20)$$

For the supersymmetric zero-energy state the exact values of  $E_h$  corresponding to  $E_- = 0$  are given by

$$E_h = (\frac{1}{2}\lambda)^{1/2}[(2n+3+2l)] - g^2/4\lambda \quad (21)$$

and the corresponding wavefunction is

$$\psi_- = c_0 \exp[-(\lambda/2)^{1/2}r^2 - gr/(2\lambda)^{1/2}]r^{l+1} \prod_{i=0}^n (1 + g_i r) \quad (22)$$

with  $g_0 = 0$ , and  $c_0$  being a normalisation constant.  $g_i$  are given by (17) and  $Z$  is not an independent quantity but is obtained in terms of  $\lambda$  and  $g$ , by eliminating  $g_i$  from (17) and (18). For example, when  $n = 0$ ,

$$Z = \frac{g}{(2\lambda)^{1/2}} l(l+1) \quad (23)$$

and, for  $n = 1$ ,

$$g_1 = \left( \frac{(l+1)g}{(2\lambda)^{1/2}} - Z \right) (l+1)^{-1} \quad (24a)$$

where from (17)  $g_1$  is given by

$$-2(l+1)g_1^2 - \frac{2g}{(2\lambda)^{1/2}} g_1 + 2a = 0. \quad (24b)$$

Eliminating  $g_1$  from (24a) and (24b) we can easily obtain  $Z$  which is given by

$$Z = \frac{g}{(2\lambda)^{1/2}} (l + \frac{3}{2}) \pm [\sqrt{2\lambda}(l+1) + g^2/8\lambda]^{1/2}. \quad (24c)$$

For general  $n$ , however, a simpler method to find  $Z$  is the following. Write  $\psi_-$  in (22) in the form

$$\psi_- = c_0 \exp[-(\frac{1}{2}\lambda)^{1/2}r^2 - gr/(2\lambda)^{1/2}]r^{l+1} \sum_{m=0}^n a_m r^m. \quad (25)$$

Then using this  $\psi_-$  in the Schrödinger equation (11) we obtain, after a few straightforward steps, the following recurrence relation:

$$\begin{aligned} a(n-m)a_m + [Z - b(m+l+2)]a_{m+1} + \frac{1}{2}(m+2)(m+2l+3)a_{m+2} &= 0 \\ m = 0, 1, 2, \dots, n \end{aligned} \quad (26)$$

where

$$a = \sqrt{2\lambda} \quad b = g/\sqrt{2\lambda}.$$

Eliminating  $a_m$  we get

$$\begin{vmatrix} Z - b(l+1) & l+1 & & & & & 0 \\ na & Z - b(l+2) & 3+2l & & & & \\ & (n-1)a & Z - b(l+3) & 6+3l & & & \\ 0 & & a & Z - b(l+n+1) & \frac{1}{2}n(n+2l+1) & & \\ & & & & & & \end{vmatrix} = 0. \quad (27)$$

This reduces to the result obtained by Bessis *et al* (1987) when  $l = 0$ .

### 3. Shifted $1/N$ expansion

The method for determining the energy eigenvalues in the shifted  $1/N$  expansion formalism is given in the paper of Imbo *et al* (1984). Hence for sake of brevity, we omit the intermediate steps and give here only the final expressions.

The energy eigenvalues in the shifted  $1/N$  expansion are given in terms of  $r_0$ , which is determined from the position of the minimum of the effective potential

$$V_{\text{eff}}(r) = \frac{1}{8r^2} + \frac{V(r)}{k^2}. \quad (28)$$

For the potential (1), with  $N = 3$ ,

$$\bar{k}^2 = 4(Zr_0 + gr_0^3 + 2\lambda r_0^4) \quad (29)$$

and the equation for determining  $r_0$  is found to be

$$(2l+1) + (2n_r+1) \left( \frac{Z+3gr_0^2+8\lambda r_0^3}{Z+gr_0^2+2\lambda r_0^3} \right)^{1/2} = 2(Zr_0 + gr_0^3 + 2\lambda r_0^4)^{1/2} \quad (30)$$

where  $n_r$  is the radial quantum number.

The final expression for the eigenvalues for the potential (1) is as follows:

$$E = \frac{\bar{k}^2}{r_0^2} \left[ \frac{1}{8} + \frac{1}{4} \frac{(-Z + gr_0^2 + \lambda r_0^3)}{(Z + gr_0^2 + 2\lambda r_0^3)} + \frac{\gamma^{(1)}}{\bar{k}^2} + \frac{\gamma^{(2)}}{\bar{k}^3} + O\left(\frac{1}{\bar{k}^4}\right) \right]. \quad (31)$$

The quantities  $\gamma^{(1)}$  and  $\gamma^{(2)}$  appearing in the corrections to the leading order of the energy expansion are as follows:

$$\gamma^{(1)} = \frac{1}{8}(1-a)(3-a) + (1+2n_r)\tilde{\epsilon}_2 + 3(1+2n_r+2n_r^2)\tilde{\epsilon}_4 - (1/\omega)[\tilde{\epsilon}_1^2 + 6(1+2n_r)\tilde{\epsilon}_1\tilde{\epsilon}_3 + (11+30n_r+30n_r^2)\tilde{\epsilon}_3^2] \quad (32)$$

$$\begin{aligned} \gamma^{(2)} = & (1+2n_r)\tilde{\delta}_2 + 3(1+2n_r+2n_r^2)\tilde{\delta}_4 + 5(3+8n_r+6n_r^2+4n_r^3)\tilde{\delta}_6 \\ & - \omega^{-1}[(1+2n_r)\tilde{\epsilon}_2^2 + 12(1+2n_r+2n_r^2)\tilde{\epsilon}_2\tilde{\epsilon}_4 \\ & + 2(21+59n_r+51n_r^2+34n_r^3)\tilde{\epsilon}_4^2 + 2\tilde{\epsilon}_1\tilde{\delta}_1 + 6(1+2n_r)\tilde{\epsilon}_1\tilde{\delta}_3 \\ & + 30(1+2n_r+2n_r^2)\tilde{\epsilon}_1\tilde{\delta}_5 + 6(1+2n_r)\tilde{\epsilon}_3\tilde{\delta}_1 \\ & + 2(11+30n_r+30n_r^2)\tilde{\epsilon}_3\tilde{\delta}_3 + 10(13+40n_r+42n_r^2+28n_r^3)\tilde{\epsilon}_3\tilde{\delta}_5] \\ & + \omega^{-2}[4\tilde{\epsilon}_1^2\tilde{\epsilon}_2 + 36(1+2n_r)\tilde{\epsilon}_1\tilde{\epsilon}_2\tilde{\epsilon}_3 + 8(11+30n_r+30n_r^2)\tilde{\epsilon}_2\tilde{\epsilon}_3^2 \\ & + 24(1+2n_r)\tilde{\epsilon}_1^2\tilde{\epsilon}_4 + 8(31+78n_r+78n_r^2)\tilde{\epsilon}_1\tilde{\epsilon}_3\tilde{\epsilon}_4 \\ & + 12(57+189n_r+225n_r^2+150n_r^3)\tilde{\epsilon}_3^2\tilde{\epsilon}_4] \\ & - \omega^{-3}[8\tilde{\epsilon}_1^3\tilde{\epsilon}_3 + 108(1+2n_r)\tilde{\epsilon}_1^2\tilde{\epsilon}_3^2 + 48(11+30n_r+30n_r^2)\tilde{\epsilon}_1\tilde{\epsilon}_3^3 \\ & + 30(31+109n_r+141n_r^2+94n_r^3)\tilde{\epsilon}_3^4] \end{aligned} \quad (33)$$

in which

$$\tilde{\epsilon}_j = \epsilon_j / (2\omega)^{j/2} \quad \tilde{\delta}_j = \delta_j / (2\omega)^{j/2}$$

$$\omega = \frac{1}{2} \left( \frac{Z + 3gr_0^2 + 8\lambda r_0^3}{Z + gr_0^2 + 2\lambda r_0^3} \right)^{1/2}$$

$$a = 2 - 2(2n_r + 1)\omega$$

$$\delta_1 = -\frac{2}{3}\delta_2 = -\frac{1}{4}(1-a)(3-a)$$

$$\delta_3 = -\frac{4}{5}\delta_4 = 2\epsilon_1 = -\frac{4}{3}\epsilon_2 = 2 - a$$

$$\epsilon_3 = -\frac{1}{2} + \frac{Z}{4(Z + gr_0^2 + 2\lambda r_0^3)}$$

$$\epsilon_4 = \frac{5}{8} - \frac{Z}{4(Z + gr_0^2 + 2\lambda r_0^3)}$$

$$\delta_5 = -\frac{3}{4} + \frac{Z}{4(Z + gr_0^2 + 2\lambda r_0^3)}$$

$$\delta_6 = \frac{7}{8} - \frac{Z}{4(Z + gr_0^2 + 2\lambda r_0^3)}$$

#### 4. Results and discussion

When  $n = 0$ , there is a single relation between  $Z$ ,  $g$  and  $\lambda$  (equation (23)). If any two of  $Z$ ,  $g$  and  $\lambda$  are fixed, the third one has a unique value given by equation (23). In this situation  $n$  is readily identified with  $n_r$ . However, when  $n > 0$ , there is a multiplicity of relations. For example, when  $n = 1$ , we see from equation (24) that for a given set of values of  $g$  and  $\lambda$  there are two solutions of  $Z$ . Similarly, for given values of  $Z$  and  $\lambda$ , there are two solutions for  $g$ . In general, for any given  $n$  and two of the parameters, there are  $(n + 1)$  possible values of the third parameter. Intuitively, one expects that these various solutions should correspond to  $n_r = 0, 1, 2, \dots, n$ . But the problem is which solution corresponds to which  $n_r$ . The identification is made possible by the shifted  $1/N$  method. We keep  $Z$  and  $\lambda$  fixed and treat  $g$  as the dependent parameter. Eigenenergies are calculated for the various sets of parameters and a comparison with the supersymmetric values identifies the value of  $n_r$ .

In tables 1 and 2 we compare the exact supersymmetric results with those obtained by the shifted  $1/N$  method. Table 1 is for  $n = 0$  and table 2 for  $n = 1$ . As noted earlier, in the latter case, for a given set of values of  $Z$  and  $\lambda$ , there are two values of  $g$ . Thus in table 2 the first and second lines refer to the same values of  $Z$  and  $\lambda$  but different values of  $g$  and  $n_r$ , and so on. These tables can be used to gauge the accuracy of the shifted  $1/N$  expansion for the potential (1) for a range of values of  $l$ ,  $Z$ ,  $\lambda$  and  $g$ .

An examination of tables 1 and 2 shows that the accuracy of the shifted  $1/N$  expansion varies considerably with the parameters. At one end of the spectrum, there is seven significant figure accuracy for some values (e.g.,  $n_r = 0$ ,  $l = 0$ ,  $Z = 10$ ,  $\lambda = 0.1$  in table 1) and, at the other, only one significant figure accuracy (e.g.,  $n_r = 1$ ,  $l = 0$ ,  $Z = 10$ ,  $\lambda = 1$  in table 2). For the two extreme cases of potential (1), i.e. the Coulomb potential and the harmonic oscillator potential, the shifted  $1/N$  expansion is known

**Table 1.** Comparison of the eigenvalues calculated from the shifted  $1/N$  expansion with the exact supersymmetric values.

$n_r$	$l$	$Z$	$\lambda$	$g$	$E(\text{shifted } 1/N \text{ method})$	$E(\text{exact, supersymmetric})$
0	0	1	0.1	0.447 21	0.171 66	0.170 82
0	1	1	0.1	0.223 61	0.993 37	0.993 03
0	2	1	0.1	0.149 07	1.509 79	1.509 69
0	3	1	0.1	0.111 80	1.981 24	1.981 21
0	0	1	1.0	1.414 21	1.627 56	1.621 32
0	1	1	1.0	0.707 11	3.411 41	3.410 53
0	2	1	1.0	0.471 40	4.894 40	4.894 19
0	3	1	1.0	0.353 55	6.332 78	6.332 71
0	0	1	10.0	4.472 14	6.226 80	6.208 20
0	1	1	10.0	2.236 07	11.057 19	11.055 34
0	2	1	10.0	1.490 71	15.597 32	15.596 92
0	3	1	10.0	1.118 03	20.093 49	20.093 36
0	0	1	100.0	14.142 14	20.753 21	20.713 20
0	1	1	100.0	7.071 07	35.233 90	35.230 34
0	2	1	100.0	4.714 05	49.442 67	49.441 92
0	3	1	100.0	3.535 53	63.608 60	63.608 36
0	0	1	1000.0	44.721 36	66.659 04	66.582 04
0	1	1	1000.0	22.360 68	111.685 01	111.678 40
0	2	1	1000.0	14.907 12	156.470 58	156.469 20
0	3	1	1000.0	11.180 34	201.215 30	201.214 87
0	0	10	0.1	4.472 14	-49.329 18	-49.329 18
0	1	10	0.1	2.236 07	-11.381 96	-11.381 97
0	2	10	0.1	1.490 71	-3.990 30	-3.990 31
0	3	10	0.1	1.118 03	-1.112 53	-1.112 54
0	0	10	1.0	14.142 14	-47.878 68	-47.878 68
0	1	10	1.0	7.071 07	-8.964 42	-8.964 47
0	2	10	1.0	4.714 05	-0.605 75	-0.605 81
0	3	10	1.0	3.535 53	3.239 03	3.238 96
0	0	10	10.0	44.721 36	-43.291 58	-43.291 80
0	1	10	10.0	22.360 68	-1.319 03	-1.319 66
0	2	10	10.0	14.907 12	10.097 63	10.096 92
0	3	10	10.0	11.180 34	17.000 07	16.999 61
0	0	10	100.0	141.421 36	-28.781 23	-28.786 80
0	1	10	100.0	70.710 68	22.862 68	22.855 34
0	2	10	100.0	47.140 45	43.945 35	43.941 92
0	3	10	100.0	35.355 34	60.516 06	60.514 61
0	0	10	1000.0	447.213 59	17.166 39	17.082 04
0	1	10	1000.0	223.606 80	99.336 72	99.303 40
0	2	10	1000.0	149.071 20	150.978 66	150.969 20
0	3	10	1000.0	111.803 40	198.124 48	198.121 12

to be very successful, but it turns out that, for some of the intermediate cases, the shifted  $1/N$  method gives rather poor results.

An eigenvalue depends on five quantities:  $n_r$ ,  $l$ ,  $Z$ ,  $\lambda$  and  $g$ . The comparison that we have made in tables 1 and 2 has an important limitation. Because  $Z$ ,  $\lambda$ ,  $g$  and  $l$  are connected by certain relations, which are different for different  $n$ , it is not possible to study the variation in the accuracy of the shifted  $1/N$  method when any four of the five quantities are kept fixed and the fifth one is varied. For this purpose we take recourse to the energies calculated by Bessis *et al* (1987). These authors have also



**Table 2.** Comparison of the eigenvalues calculated from the shifted  $1/N$  expansion with the exact supersymmetric values. Components of each pair of lines (starting with the first line) refer to the same values of  $l, Z$  and  $\lambda$  but different values of  $g$  and  $n_r$ .

$n_r$	$l$	$Z$	$\lambda$	$g$	$E(\text{shifted } 1/N \text{ method})$	$E(\text{exact, supersymmetric})$
1	0	1	0.1	0.096 20	1.158 40	1.094 90
0				0.574 62	0.293 49	0.292 56
1	1	1	0.1	0.009 69	1.579 47	1.565 01
0				0.362 98	1.236 19	1.235 86
1	2	1	0.1	-0.020 25	2.016 32	2.011 44
0				0.281 13	1.814 97	1.814 88
1	3	1	0.1	-0.033 59	2.458 93	2.456 85
0				0.234 84	2.321 84	2.321 80
1	0	1	1.0	-0.179 99	3.654 88	3.527 44
0				2.301 31	2.217 96	2.211 53
1	1	1	1.0	-0.388 85	4.939 66	4.911 95
0				1.567 37	4.336 45	4.335 59
1	2	1	1.0	-0.430 48	6.326 97	6.317 63
0				1.255 44	5.970 13	5.969 93
1	3	1	1.0	-0.434 75	7.734 88	7.730 92
0				1.071 15	7.491 40	7.491 33
1	0	1	10.0	-3.426 12	11.129 19	10.886 88
0				10.134 32	8.631 16	8.612 73
1	1	1	10.0	-3.609 56	15.379 90	15.326 75
0				7.336 34	14.308 78	14.306 93
1	2	1	10.0	-3.428 01	19.848 92	19.830 83
0				6.036 75	19.213 96	19.213 55
1	3	1	10.0	-3.224 73	24.344 50	24.336 77
0				5.237 19	23.911 18	23.911 04
1	0	10	0.1	2.216 24	-11.163 53	-11.161 31
0				4.491 96	-49.326 23	-49.326 23
1	1	10	0.1	1.452 66	-3.709 88	-3.710 27
0				2.274 12	-11.363 86	-11.363 86
1	2	10	0.1	1.065 45	-0.822 41	-0.825 51
0				1.543 29	-3.941 92	-3.941 92
1	3	10	0.1	0.831 91	0.733 23	0.729 50
0				1.180 55	-1.024 59	-1.024 59
1	0	10	1.0	6.876 43	-8.284 23	-8.285 77
0				14.336 78	-47.850 26	-47.850 27
1	1	10	1.0	4.365 57	0.224 22	0.185 20
0				7.419 55	-8.812 62	-8.812 67
1	2	10	1.0	3.097 98	3.996 76	3.964 58
0				5.151 59	-0.270 71	-0.270 77
1	3	10	1.0	2.350 91	6.415 85	6.396 48
0				4.013 06	3.752 09	3.752 02
1	0	10	10.0	20.513 30	1.093 44	0.660 45
0				46.568 73	-43.035 56	-43.035 84
1	1	10	10.0	12.023 07	12.294 50	12.038 62
0				25.244 73	-0.279 31	-0.279 94
1	2	10	10.0	7.961 12	18.650 74	18.540 12
0				18.126 34	11.911 24	11.910 51
1	3	10	10.0	5.687 54	23.839 52	23.788 04
0				14.437 07	19.386 49	19.386 02

employed the method of Stieltjes moments to generate rapidly converging upper and lower bounds to the ground-state energy  $E_0(Z, g, \lambda)$  for arbitrary  $g$ ,  $Z$  and positive  $\lambda$ . The resulting values are accurate to 9 and 10 significant figures. In table 3 we compare the ground-state energy values obtained by the shifted  $1/N$  method with those of Bessis *et al* (1987) for  $Z = 1$ ,  $\lambda = 1$  and a series of  $g$  values. It will be noticed that as we go down this table, the error reaches a maximum at  $g \approx -0.5$  and then it steadily decreases as  $g$  assumes greater positive values. For very large  $g$ , the potential (1) tends to the linear potential; the trend of results in table 3 indicates that for this potential the shifted  $1/N$  expansion should give very good results for the ground state. This is in agreement with the findings of Imbo *et al* (1984).

**Table 3.** Comparison between the eigenvalues calculated from the shifted  $1/N$  expansion and those of Bessis *et al* (1987).

$g$	$E$ (Shifted $1/N$ method)	$E$ (Bessis <i>et al</i> 1987)	Percentage difference
-2.0	-1.149 33	-1.171 7	1.9
-1.0	-0.218 57	-0.226 19	3.4
-0.5	0.202 84	0.196 00	3.5
-0.1	0.522 47	0.515 94	1.3
0.0	0.600 25	0.593 77	1.1
0.1	0.677 25	0.670 81	1.0
0.5	0.977 90	0.971 62	0.6
1.0	1.339 1	1.332 8	0.5
2.0	2.021 3	2.014 9	0.3
3.0	2.660 8	2.654 1	0.3
5.0	3.845 0	3.837 4	0.2

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